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CHARACTERISTICS OF CONSTITUTIVE RELATIONS IN SOIL PLASTICITY FOR UNDRAINED BEHAVIOR

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Abstract—Elastic–plastic relations which are subjected to the constraint of incompressibility, that is pertinent to undrained behavior of soils, are assessed and compared with the relations for drained behavior. The tangent relations for mixed control variables (total stress and strain components) are established explicitly. The criterion signalling plastic loading, and elastic unloading, is considered in particular. A criterion for unique response, as expressed in terms of elastic and plastic loading, is formulated as a condition on the particular plastic modulus that corresponds to undrained behavior. Stability is discussed in terms of appropriate critical values of the plastic modulus. It appears that, for a frictional material, dilatancy has a stabilizing effect whereas contractancy may be destabilizing compared to drained behavior.

INTRODUCTION

The behavior of an elastic–plastic material is governed by bilinear tangent modulus relations, and the forms of these relations depend on whether plastic loading or elastic unloading takes place. While the tangent stiffness and compliance relations in plastic loading depend only on the actual values of the state variables (stress and/or strain components and hardening variables), the loading/unloading criterion depends also on the choice of control variables (stress and/or strain) components and involves the incremental change of these variables. The control variables are those mixed stress and strain components which are prescribed (at a certain instant) and are used as input to the constitutive relation. The response variables, which are energy-conjugated to the control variables, are thus obtained as output from the constitutive relation via integration along a given loading/unloading path. A few examples in two-invariant stress space, for drained as well as undrained behavior and for a specific elastic–plastic model, were discussed by Mroz *et al.* (1979).

It is clear that the concept of control and response variables has a meaning in conjunction with the treatment of boundary value problems only if the state is homogeneous. This is traditionally assumed for a specimen in laboratory experiments, such as plane strain and conventional triaxial tests, prior to bifurcation and the development of localized deformation modes. The entire discussion in this paper refers to such homogeneous, that is, constitutive behavior.

A quite general analysis of the consequences of the particular choice of state and control variables for drained behavior was presented by Klisinski *et al.* (1991), and we shall briefly restate some results under the specific assumption that the yield surface is represented in stress space. Then we shall focus on undrained behavior, which is pertinent to soil behavior when drainage through the boundary of a soil body is prevented. In this paper undrained behavior is taken as synonymous with complete (pointwise) incompressibility regardless of the magnitude of the mean effective stress, cf. Mroz *et al.* (1979). This assumption is quite realistic since the bulk modulus of water is nearly two or three magnitudes higher than that of the soil skeleton. When the permeability is large, such as for sand, it is clear that pointwise undrained behavior cannot be ensured unless the state is quite homogeneous throughout the body under consideration. Strain localization under undrained behavior was discussed by Rice (1975), Rice and Cleary (1976) and Vardoulakis (1985).

It is intuitively clear that the tangent compliance matrix must become singular due to the incompressibility condition. In other words, it is anticipated that this condition causes the deformation to vanish for a certain mode of change in the applied total stresses, i.e. the material locks. Via a spectral analysis of the tangent compliance relation it will also be shown in this paper that the stiffness is generally larger in the undrained than in the drained situation. The specific requirements on the actual plasticity model for which a stabilizing effect is obtained from the incompressibility constraint are also discussed.

The developed tangent relations do not only have a didactic value but have practical significance in terms of a step-by-step procedure for truly finite increments. Such an explicit integration technique is natural (but not necessarily the most efficient) within the framework of a **mixed** finite element method, the description of which is outside the scope of this paper. The incompressibility condition is then incorporated in a "strong", i.e. exact, fashion. In the more conventional **coupled** finite element formulation this condition is invoked in a "weak" sense via a variational statement.

Matrix notation will be used throughout the paper. Second rank tensors are represented by column vectors (such as the effective stress vector $\boldsymbol{\sigma}$), while fourth rank tensors are represented by square matrices (such as the tangential compliance matrix for undrained behavior, that is denoted $\mathbf{C}_u$). Stresses and strains are taken to be positive in compression, which differs from the conventional notation in continuum mechanics but conforms to common practice in soil mechanics.

CONTROL VARIABLES—LOADING CONDITION IN PLASTICITY

In the theory of small deformations it is assumed that the total strain rate is decomposed additively into elastic and plastic rates

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{e} + \dot{\boldsymbol{\varepsilon}}^{p}. \tag{1}$$

The elastic part is given by the tangential relationships

$$\dot{\boldsymbol{\varepsilon}}^{e} = \mathbf{C}^{e}\dot{\boldsymbol{\sigma}}, \quad \dot{\boldsymbol{\sigma}} = \mathbf{D}^{e}\dot{\boldsymbol{\varepsilon}}^{e} \tag{2}$$

where $\mathbf{C}^e$ and $\mathbf{D}^e = (\mathbf{C}^e)^{-1}$ are the matrices of tangential elastic compliance and stiffness moduli respectively. We shall assume subsequently, without losing the essential features, that $\mathbf{C}^e$ and $\mathbf{D}^e$ are constant matrices. Via the well-known effective stress principle in soil mechanics, the effective stress $\boldsymbol{\sigma}$ is related to the total stress $\mathbf{s}$ and the excess pore pressure u as

$$\dot{\boldsymbol{\sigma}} = \dot{\mathbf{s}} - \dot{u}\boldsymbol{\delta} \tag{3}$$

where $\boldsymbol{\delta}$ is a vector representation of Kronecker's delta, and where it has been assumed that the stresses are positive in compression. It is assumed that the plastically admissible stress states $\boldsymbol{\sigma}$ are contained in the convex set B

$$B = \{\boldsymbol{\sigma} | F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leqslant 0\} \tag{4}$$

where $F(\boldsymbol{\sigma}, \boldsymbol{\kappa})$ is the yield function and $F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = 0$ represents the state boundary surface. In order to allow for hardening/softening of the current yield surface, we have introduced the column vector $\boldsymbol{\kappa}$ representing components of the hardening/softening internal variables.

Needless to say, elastic states are associated with $F < 0$, while plastic states are defined by $F = 0$.

The direction of $\dot{\boldsymbol{\varepsilon}}^p$ is determined by the direction of a vector $\mathbf{m}$ which defines a general non-associated flow rule

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\mathbf{m} \tag{5}$$

where $\dot{\lambda} \geqslant 0$ is a scalar plastic multiplier. The evolution rule for $\boldsymbol{\kappa}$ may, for a quite general class of hardening rules, be postulated to be of the form

$$\dot{\boldsymbol{\kappa}} = \mathbf{h}(\dot{\boldsymbol{\varepsilon}}^p) = \dot{\lambda}\mathbf{h}(\mathbf{m}) \tag{6}$$

where it is implied that $\mathbf{h}$ is a first degree homogeneous function. This homogeneity is necessary in order to give a bilinear tangent modulus relation between the control and response variables.

The well-known plastic loading criteria that apply at a plastic state, i.e. when $F = 0$, are as follows:

$$\begin{aligned} &\dot{\lambda} > 0, \quad \dot{F} = 0, \quad \text{plastic loading (P)} \\ &\dot{\lambda} = 0, \quad \dot{F} = 0, \quad \text{neutral loading (N)} \\ &\dot{\lambda} = 0, \quad \dot{F} < 0, \quad \text{elastic unloading (E).} \end{aligned} \tag{7}$$

These loading-unloading criteria can be summarized in the form of the Kuhn-Tucker conditions:

$$\dot{\lambda} \geqslant 0, \quad \dot{F} \leqslant 0, \quad \dot{\lambda}\dot{F} = 0. \tag{8}$$

These conditions are completely general with respect to the choice of control variables and/or linear constraints that might be imposed on the stresses and strains.

While either $\dot{\boldsymbol{\varepsilon}}$ or $\dot{\boldsymbol{\sigma}}$ may be chosen as control variables for drained ($u = 0, \boldsymbol{\sigma} = \mathbf{s}$) or partly drained ($u \neq 0, \boldsymbol{\sigma} \neq \mathbf{s}$) behavior, it appears that not all strain components can be chosen as control variables under undrained conditions. The reason is that the undrained condition is assumed (see the Introduction) to be represented by the linear incompressibility constraint

$$\dot{\varepsilon}_v = \boldsymbol{\delta}^T\dot{\boldsymbol{\varepsilon}} = 0 \tag{9}$$

where ε_v is the volumetric strain. Complete stress control is defined by prescribing $\mathbf{s}$, which gives $\boldsymbol{\varepsilon}$ and u as response variables.

As mentioned above, complete strain control is not possible for undrained behavior. However, mixed control in terms of a combination of stress and strain components (or rather their rates) is commonly adopted. For example, in conventional undrained triaxial compression tests, the horizontal (confining) stress and the vertical strain are often chosen as control variables, which immediately leaves the horizontal strain to be uniquely defined by the incompressibility condition.

Subsequently we shall need the appropriate elastic tangent relationship between the chosen control and response variables. If $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are decomposed into the energy conjugate portions $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$ respectively, i.e.

$$\boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \end{bmatrix} \tag{10}$$

then the relations (2) may be expressed as

$$\begin{bmatrix}\dot{\varepsilon}_1^e\\ \dot{\varepsilon}_2^e\end{bmatrix}=\begin{bmatrix}\mathbf{C}_{11}^e & \mathbf{C}_{12}^e\\ (\mathbf{C}_{12}^e)^T & \mathbf{C}_{22}^e\end{bmatrix}\begin{bmatrix}\dot{\sigma}_1\\ \dot{\sigma}_2\end{bmatrix}\quad \begin{bmatrix}\dot{\sigma}_1\\ \dot{\sigma}_2\end{bmatrix}=\begin{bmatrix}\mathbf{D}_{11}^e & \mathbf{D}_{12}^e\\ (\mathbf{D}_{12}^e)^T & \mathbf{D}_{22}^e\end{bmatrix}\begin{bmatrix}\dot{\varepsilon}_1^e\\ \dot{\varepsilon}_2^e\end{bmatrix}. \tag{11}$$

Assuming that $\mathbf{s}_1$ and $\boldsymbol{\varepsilon}_2$ are the control variables, whereas $\boldsymbol{\varepsilon}_1$ and $\mathbf{s}_2$ are the response variables, we may partially invert eqns (11) to obtain the relationship

$$\begin{bmatrix}\dot{\varepsilon}_1^e\\ \dot{\sigma}_2\end{bmatrix}=\begin{bmatrix}\mathbf{E}_{11}^e & \mathbf{E}_{12}^e\\ \mathbf{E}_{21}^e & \mathbf{E}_{22}^e\end{bmatrix}\begin{bmatrix}\dot{\sigma}_1\\ \dot{\varepsilon}_2^e\end{bmatrix} \tag{12}$$

where

$$\begin{aligned}
\mathbf{E}_{11}^e &= \mathbf{C}_{11}^e-\mathbf{C}_{12}^e(\mathbf{C}_{22}^e)^{-1}(\mathbf{C}_{12}^e)^T = (\mathbf{D}_{11}^e)^{-1}\\
\mathbf{E}_{12}^e &= \mathbf{C}_{12}^e(\mathbf{C}_{22}^e)^{-1} = -(\mathbf{D}_{11}^e)^{-1}\mathbf{D}_{12}^e\\
\mathbf{E}_{21}^e &= -(\mathbf{E}_{12}^e)^T\\
\mathbf{E}_{22}^e &= (\mathbf{C}_{22}^e)^{-1} = \mathbf{D}_{22}^e-(\mathbf{D}_{12}^e)^T(\mathbf{D}_{11}^e)^{-1}\mathbf{D}_{12}^e.
\end{aligned} \tag{13}$$

Since $\mathbf{E}_{11}^e = (\mathbf{D}_{11}^e)^{-1}$ and $\mathbf{E}_{22}^e = (\mathbf{C}_{22}^e)^{-1}$ are positive definite, it can simply be shown that $\mathbf{E}^e$ is positive definite.

TANGENT RELATIONSHIPS UNDER DRAINED CONDITIONS

Stress and strain control

For completeness, we shall briefly derive the pertinent equations for drained behavior (and partly drained behavior, when u is known), whereby we follow closely Klisinski *et al.* (1990). The consistency condition in a plastic state is, with the notation $\mathbf{n} = \partial F/\partial \boldsymbol{\sigma}$

$$\dot{F} = \mathbf{n}^T\dot{\boldsymbol{\sigma}}+\left(\frac{\partial F}{\partial \boldsymbol{\kappa}}\right)^T\dot{\boldsymbol{\kappa}} \leqslant 0 \tag{14}$$

which with the flow rule eqn (5) and the hardening rule eqn (6) can be rewritten as

$$\dot{F} = \mathbf{n}^T\dot{\boldsymbol{\sigma}}-H\dot{\lambda} \leqslant 0 \tag{15}$$

where H is the generalized plastic modulus

$$H = -\left(\frac{\partial F}{\partial \boldsymbol{\kappa}}\right)^T\mathbf{h}(\mathbf{m}). \tag{16}$$

Under stress control we can directly use eqn (15). The loading criteria (7) are then equivalent with the conditions

$$\dot{\lambda} = \frac{1}{H}\mathbf{n}^T\dot{\boldsymbol{\sigma}} > 0 \quad (\mathrm{P}) \tag{17}$$

$$\mathbf{n}^T\dot{\boldsymbol{\sigma}} = 0 \quad (\mathrm{N}) \tag{18}$$

$$\mathbf{n}^T\dot{\boldsymbol{\sigma}} < 0 \quad (\mathrm{E}) \tag{19}$$

and it is clear that the conditions in (17), (18) and (19) are unambiguous only if $H > 0$. This requirement is thus necessary and sufficient in order to ensure a unique drained response under stress control. The consequent compliance modulus relation reads

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{C}\dot{\boldsymbol{\sigma}} = \mathbf{C}\dot{\mathbf{s}} \quad (\dot{u} = 0) \tag{20}$$

where $\mathbf{C}$ is the conventional tangent compliance matrix

$$\mathbf{C} = \mathbf{C}^e + \frac{1}{H}\mathbf{m}\mathbf{n}^T \quad \text{when} \quad F = 0, \quad \mathbf{n}^T\dot{\mathbf{s}} > 0. \tag{21}$$

Under strain control, on the other hand, $\dot{\boldsymbol{\sigma}}$ has to be expressed in terms of $\dot{\boldsymbol{\varepsilon}}$ via eqn (2), whereby eqn (15) is replaced by

$$\dot{F} = \mathbf{n}^T\mathbf{D}^e\dot{\boldsymbol{\varepsilon}} - K\dot{\lambda} \leqslant 0 \tag{22}$$

where K is defined by

$$K = H + \mathbf{n}^T\mathbf{D}^e\mathbf{m}. \tag{23}$$

Similarly to the previous situation, $\dot{\lambda}$ may be solved from eqn (22) when $\dot{F} = 0$

$$\dot{\lambda} = \frac{1}{K}\mathbf{n}^T\mathbf{D}^e\dot{\boldsymbol{\varepsilon}} \tag{24}$$

and the loading criteria will be unambiguous only if $K > 0$. The pertinent stiffness modulus relation is obtained with eqns (2) and (24)

$$\dot{\mathbf{s}} = \mathbf{D}\dot{\boldsymbol{\varepsilon}} \quad (\dot{u} = 0) \tag{25}$$

where $\mathbf{D}$ is the tangent stiffness matrix

$$\mathbf{D} = \mathbf{D}^e - \frac{1}{K}\mathbf{D}^e\mathbf{m}\mathbf{n}^T\mathbf{D}^e \quad \text{when} \quad F = 0, \quad \mathbf{n}^T\mathbf{D}^e\dot{\boldsymbol{\varepsilon}} > 0. \tag{26}$$

Mixed control

When $\mathbf{s}_1(=\boldsymbol{\sigma}_1)$ and $\boldsymbol{\varepsilon}_2$ are chosen as the control variables it is necessary to express $\dot{\boldsymbol{\sigma}}_2$ in terms of these control variables via the elastic tangent relationship (12). The consistency condition (15) then becomes, with the obvious notation $\mathbf{n}_1 = \partial F/\partial \boldsymbol{\sigma}_1$ and $\mathbf{n}_2 = \partial F/\partial \boldsymbol{\sigma}_2$,

$$\dot{F} = \mathbf{n}_1^T\dot{\boldsymbol{\sigma}}_1 + \mathbf{n}_2^T\dot{\boldsymbol{\sigma}}_2 - H\dot{\lambda} = \phi - K\dot{\lambda} \leqslant 0 \tag{27}$$

where ϕ is the loading function

$$\phi = \hat{\mathbf{n}}_1^T\dot{\boldsymbol{\sigma}}_1 + \hat{\mathbf{n}}_2^T\dot{\boldsymbol{\varepsilon}}_2 \tag{28}$$

and K is the generalized plastic modulus under mixed control

$$K = H + \mathbf{n}_2^T\mathbf{E}_{22}^e\mathbf{m}_2. \tag{29}$$

We have introduced the "transformed" gradients of F under mixed control:

$$\hat{\mathbf{n}}_1 = \mathbf{n}_1 - \mathbf{E}_{12}^e\mathbf{n}_2, \quad \hat{\mathbf{n}}_2 = \mathbf{E}_{22}^e\mathbf{n}_2. \tag{30}$$

The loading criteria become

$$\dot{\lambda} = \frac{1}{K}\phi > 0 \quad (\mathrm{P}) \tag{31}$$

$$\phi = 0 \quad (\mathrm{N}) \tag{32}$$

$$\phi < 0 \quad (\mathrm{E}) \tag{33}$$

and the requirement for unambiguous criteria is, again, $K > 0$. The pertinent tangent relationship at plastic loading, i.e. when $F = 0$ and $\phi > 0$, is obtained with eqns (12) and (31) as

$$\begin{bmatrix} \dot{\boldsymbol{\varepsilon}}_1 \\ \dot{\mathbf{s}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{s}}_1 \\ \dot{\boldsymbol{\varepsilon}}_2 \end{bmatrix}. \tag{34}$$

Upon introducing the "transformed" flow vectors $\hat{\mathbf{m}}_1$ and $\hat{\mathbf{m}}_2$ from $\mathbf{m}_1$ and $\mathbf{m}_2$ as in eqn (30), we may express the tangent matrix $\mathbf{E}$ as

$$\begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{E}^e_{11} & \mathbf{E}^e_{12} \\ \mathbf{E}^e_{21} & \mathbf{E}^e_{22} \end{bmatrix} + \frac{1}{K} \begin{bmatrix} \hat{\mathbf{m}}_1\hat{\mathbf{n}}_1^T & \hat{\mathbf{m}}_1\hat{\mathbf{n}}_2^T \\ -\hat{\mathbf{m}}_2\hat{\mathbf{n}}_1^T & -\hat{\mathbf{m}}_2\hat{\mathbf{n}}_2^T \end{bmatrix}. \tag{35}$$

It may simply be checked that the special cases of stress and strain control directly follow from the present general case of mixed control by, respectively, the identifications $\dot{\mathbf{s}}_1 = \dot{\mathbf{s}}(\dot{\boldsymbol{\varepsilon}}_1 = \dot{\boldsymbol{\varepsilon}})$ and $\dot{\boldsymbol{\varepsilon}}_2 = \dot{\boldsymbol{\varepsilon}}(\dot{\mathbf{s}}_2 = \dot{\mathbf{s}})$.

In the next section we shall derive the corresponding relations under the constraint of incompressibility pertinent to undrained behavior.

TANGENT RELATIONSHIPS UNDER UNDRAINED CONDTIONS

Total stress control

It is possible to choose the total stress $\mathbf{s}$ as the control variable under the constraint of $\dot{\varepsilon}_v$ being prescribed. From eqns (2), (3), (5) and (9) we obtain

$$\dot{\varepsilon}_v = \boldsymbol{\delta}^T\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\delta}^T(\dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p) = \bar{\boldsymbol{\delta}}^T\dot{\mathbf{s}} - c^{-1}\dot{u} + \dot{\lambda}m_v = 0 \tag{36}$$

where we have introduced the notations $\bar{\boldsymbol{\delta}} = \mathbf{C}^e\boldsymbol{\delta}$, $c = 1/\boldsymbol{\delta}^T\mathbf{C}^e\boldsymbol{\delta}$ and $m_v = \boldsymbol{\delta}^T\mathbf{m}$. From eqn (36) we can solve for $\dot{u}$

$$\dot{u} = c(\bar{\boldsymbol{\delta}}^T\dot{\mathbf{s}} + \dot{\lambda}m_v). \tag{37}$$

Inserting eqn (37) into the effective stress principle eqn (3) gives

$$\dot{\boldsymbol{\sigma}} = \dot{\mathbf{s}} - \dot{u}\boldsymbol{\delta} = \mathbf{D}^e\mathbf{C}^e_u\dot{\mathbf{s}} - \dot{\lambda}cm_v\boldsymbol{\delta} \tag{38}$$

where $\mathbf{C}^e_u$ is the singular (as proven later) "undrained elastic compliance matrix"

$$\mathbf{C}^e_u = \mathbf{C}^e - c\bar{\boldsymbol{\delta}}\bar{\boldsymbol{\delta}}^T. \tag{39}$$

This matrix is obtained by imposing the incompressibility condition within the elastic range. Inserting $\dot{\boldsymbol{\sigma}}$ from eqn (38) into the consistency condition eqn (15) for plastic loading now gives

$$\dot{F} = \mathbf{n}^T\dot{\boldsymbol{\sigma}} - H\dot{\lambda} = \mathbf{n}^T\mathbf{D}^e\mathbf{C}^e_u\dot{\mathbf{s}} - \mathrm{H}_u\dot{\lambda} = 0. \tag{40}$$

We may thus solve for the plastic multiplier

$$\dot{\lambda} = \frac{1}{H_u}\phi_u \tag{41}$$

where H_u is the "undrained plastic modulus"

$$H_u = H + cn_v m_v, \quad n_v = \boldsymbol{\delta}^T\mathbf{n} \tag{42}$$

and ϕ_u is the "undrained loading function"

$$\phi_u = \mathbf{n}^T\mathbf{D}^e\mathbf{C}_u^e\dot{\mathbf{s}} = \mathbf{n}_u^T\dot{\mathbf{s}}. \tag{43}$$

From the definition of $\mathbf{C}_u^e$ in (39) we have obtained the "undrained gradient" $\mathbf{n}_u$ as

$$\mathbf{n}_u = \mathbf{n} - cn_v\boldsymbol{\delta} \tag{44}$$

which is a purely deviatoric tensor, i.e. its volumetric part is zero.

Similarly to the case of drained behavior, we conclude that $H_u > 0$ is the proper requirement for unique response, in which case $\phi_u > 0$ signals plastic loading.

Inserting $\dot{\lambda}$ from eqn (41) into the expression for $\dot{u}$ in eqn (37), we obtain for $\phi_u > 0$

$$\dot{u} = c\left(\boldsymbol{\delta} + \frac{1}{H_u}m_v\mathbf{n}_u\right)^T\dot{\mathbf{s}}. \tag{45}$$

Combining this expression with the flow rule eqn (5) we obtain, finally, the response variable $\dot{\boldsymbol{\varepsilon}}$ via the compliance relationship

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{C}_u\dot{\mathbf{s}} \tag{46}$$

where $\mathbf{C}_u$ is the undrained tangent compliance matrix

$$\mathbf{C}_u = \mathbf{C}_u^e + \frac{1}{H_u}\mathbf{m}_u\mathbf{n}_u^T \tag{47}$$

and $\mathbf{m}_u$ is defined like $\mathbf{n}_u$ in eqn (44), i.e.

$$\mathbf{m}_u = \mathbf{m} - cm_v\boldsymbol{\delta}. \tag{48}$$

In the case of elastic unloading, $\phi_u \leqslant 0$, we obtain (since $\dot{\lambda} = 0$)

$$\dot{u} = c\boldsymbol{\delta}^T\dot{\mathbf{s}} = \dot{s}_m + c\boldsymbol{\delta}^T\dot{\mathbf{s}}_d \tag{49}$$

where d denotes deviator, i.e. $\mathbf{s}_d = \mathbf{s} - s_m\boldsymbol{\delta}$ and $s_m = \boldsymbol{\delta}^T\mathbf{s}/3$. Moreover, we obtain

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{C}_u^e\dot{\mathbf{s}} \tag{50}$$

which clearly shows that $\mathbf{C}_u^e$ is, in fact, the proper elastic modulus matrix for undrained behavior.

Let us now consider a few important special cases:

Associated flow rule. In the case of an associated flow rule, $\mathbf{m} = \mathbf{n}$, we obtain from (45) and (47) for $\phi_u > 0$

$$\dot{u} = c(\bar{\delta} + \frac{1}{H_u} n_v \mathbf{n}_u)^T \dot{\mathbf{s}} \tag{51}$$

$$\mathbf{C}_u = \mathbf{C}_u^e + \frac{1}{H_u} \mathbf{n}_u \mathbf{n}_u^T \tag{52}$$

where

$$H_u = H + cn_v^2. \tag{53}$$

As in the drained case, $\mathbf{C}_u$ becomes symmetrical.

Incompressible plastic flow. We now return to the general case of non-associated plasticity defined by $\mathbf{m} \neq \mathbf{n}$. An important special case is incompressible plastic flow, $\dot{\varepsilon}_v^p = 0$ or $m_v = 0$, which is pertinent to cohesive materials for which the plastic potential is pressure-independent. We obtain for $\phi_u > 0$

$$\dot{u} = c\bar{\delta}^T \dot{\mathbf{s}} \tag{54}$$

and

$$\mathbf{C}_u = \mathbf{C}_u^e + \frac{1}{H} \mathbf{m} \mathbf{n}_u^T \tag{55}$$

where the fact that $H_u = H$ was used.

It is noted that, despite the absence of compressibility in the plastic portion of the strain rate, there is still a coupling between the incompressibility condition and the plastic response that remains apparent in the total response. We also note that the excess pore pressure development is identical to that for elastic response. However, for a completely associated flow rule, i.e. when $\mathbf{m} = \mathbf{n}$, $n_v = 0$, eqn (55) is further simplified to

$$\mathbf{C}_u = \mathbf{C}_u^e + \frac{1}{H} \mathbf{n}\mathbf{n}^T \tag{56}$$

subjected to the loading criterion $\phi_u = \phi = \mathbf{n}^T \dot{\mathbf{s}} > 0$, which reduces to that of drained behavior. In this particular case it is clear that the incompressibility condition only affects the elastic compliance matrix $\mathbf{C}_u^e$.

Isotropic elasticity. The final special case is that the elastic response is linear and isotropic, which is defined by

$$\mathbf{C}^e = \frac{1}{2G}(\mathbf{I} - \tfrac{1}{3}\delta\delta^T) + \frac{1}{9K}\delta\delta^T \tag{57}$$

where G and K are the shear and bulk modulus respectively. This gives $\bar{\delta} = \delta/3K$ and $c = K$. Equation (39) gives

$$\mathbf{C}_u^e = \frac{1}{2G}(\mathbf{I} - \tfrac{1}{3}\delta\delta^T) \tag{58}$$

and it is simple to see that this matrix is singular.

Moreover, we obtain

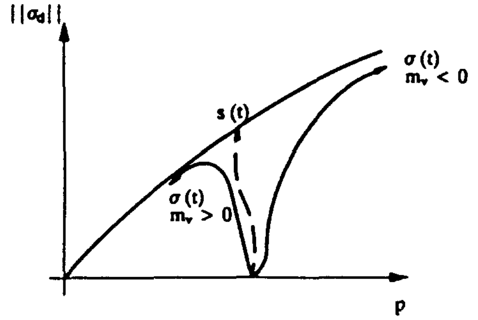

Fig. 1. Effective stress paths at undrained loading for contractant and dilatant behavior.

$$H_u = H + Kn_v m_v \tag{59}$$

$$\mathbf{n}_u = \mathbf{n}_d = \mathbf{n} - n_v \boldsymbol{\delta}/3, \quad \mathbf{m}_u = \mathbf{m}_d. \tag{60}$$

From eqns (37) and (41), or eqn (45) we obtain the development of excess pore pressure

$$\dot{u} = \dot{s}_m + Km_v\dot{\lambda} = \dot{s}_m + \frac{K}{H_u} m_v \phi_u \tag{61}$$

and the development of effective pressure (mean effective stress) defined as $p = \sigma_m (= \boldsymbol{\delta}^T \boldsymbol{\sigma}/3)$

$$\dot{p} = \dot{\sigma}_m = \dot{s}_m - \dot{u} = -Km_v\dot{\lambda} = -\frac{K}{H_u} m_v \phi_u \tag{62}$$

where

$$\phi_u = \mathbf{n}_d^T \dot{\mathbf{s}}. \tag{63}$$

The different characteristic behavior that can be extracted from eqn (61) is shown in Fig. 1 for an arbitrarily applied stress path $\mathbf{s}(t)$.

Consider a frictional material that is characterized by $n_v < 0$, i.e. the yield surface is of the "cone" type. From eqn (59) it follows that dilatant behavior ($m_v < 0$) implies that $H_u > H$, while contractant behavior ($m_v > 0$) implies that $H_u < H$. When the hardening H of the drained material has decreased to a value $-Kn_v m_v > 0$, we can expect that the response of the contractant material will move into the softening regime ($H_u < 0$), while the dilatant material still hardens and continues to do so even when $H < 0$. For a cohesive material, on the other hand, that is defined by $n_v = 0$, we shall always have $H_u = H$ and the hardening modulus is thus unaffected by the incompressibility condition. These predictions are in good agreement with experimental findings.

Based on a simplified frictional model for an undrained layer, the dilatant hardening effect was shown by Rice (1975), who calculated the tangent compliance modulus corresponding to applied shear stress while the normal stress was held constant. In other words, a diagonal term of $\mathbf{C}_u$ was calculated.

In the case of an associated flow rule, $H_u = H + Kn_v^2$, we conclude that $H_u \geqslant H$ always and the undrained condition has a stabilizing effect independent of whether the behavior is dilatant or contractant. This is the situation for the classical Cam–Clay models.

Finally, we shall comment on the loading criterion $\phi_u > 0$, where ϕ_u is given in (63). It appears that plastic loading can be judged entirely from the deviatoric portion of $\dot{\mathbf{s}}$, whereby this criterion resembles the drained behavior of a cohesive material in terms of the effective stress rate $\dot{\boldsymbol{\sigma}}$.

Mixed control

Let us choose $\mathbf{s}_1$ and $\boldsymbol{\varepsilon}_2$ as control variables, whereby u, $\boldsymbol{\varepsilon}_1$ and $\mathbf{s}_2$ become the response variables. We then obtain

$$\dot{\varepsilon}_v = \boldsymbol{\delta}_1^T \dot{\boldsymbol{\varepsilon}}_1 + \boldsymbol{\delta}_2^T \dot{\boldsymbol{\varepsilon}}_2 = 0. \tag{64}$$

From the decompositions

$$\begin{bmatrix} \dot{\boldsymbol{\sigma}}_1 \\ \dot{\boldsymbol{\sigma}}_2 \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{s}}_1 \\ \dot{\mathbf{s}}_2 \end{bmatrix} - \dot{u} \begin{bmatrix} \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \end{bmatrix}, \quad \begin{bmatrix} \dot{\boldsymbol{\varepsilon}}_1^p \\ \dot{\boldsymbol{\varepsilon}}_2^p \end{bmatrix} = \dot{\lambda} \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix} \tag{65}$$

and the elastic tangent relationship (12), we may express the condition $\dot{\varepsilon}_v = 0$ in (64) as

$$\dot{\varepsilon}_v = \bar{\boldsymbol{\delta}}_1^T \dot{\mathbf{s}}_1 + \bar{\boldsymbol{\delta}}_2^T \dot{\boldsymbol{\varepsilon}}_2 - c_1^{-1} \dot{u} + \dot{\lambda} \hat{m}_{1v} = 0 \tag{66}$$

where we have introduced the adjusted Kronecker deltas $\bar{\boldsymbol{\delta}}_1$ and $\bar{\boldsymbol{\delta}}_2$ as

$$\bar{\boldsymbol{\delta}}_1 = \mathbf{E}_{11}^c \boldsymbol{\delta}_1, \quad \bar{\boldsymbol{\delta}}_2 = \boldsymbol{\delta}_2 - \mathbf{E}_{21}^c \boldsymbol{\delta}_1 \tag{67}$$

and $\hat{m}_{1v} = \boldsymbol{\delta}_1^T \hat{\mathbf{m}}_1$, where $\hat{\mathbf{m}}_1$ and $\hat{\mathbf{m}}_2$ are still given as in eqn (30). Furthermore, we have used the notation

$$c_1 = 1/\boldsymbol{\delta}_1^T \mathbf{E}_{11}^c \boldsymbol{\delta}_1. \tag{68}$$

We may now solve for $\dot{u}$ from eqn (66) to obtain

$$\dot{u} = c_1 (\bar{\boldsymbol{\delta}}_1^T \dot{\mathbf{s}}_1 + \bar{\boldsymbol{\delta}}_2^T \dot{\boldsymbol{\varepsilon}}_2 + \dot{\lambda} \hat{m}_{1v}) \tag{69}$$

where the plastic multiplier $\dot{\lambda}$ is still to be calculated from the consistency condition at plastic loading. Inserting eqn (69) into the effective stress principle (65) gives

$$\dot{\boldsymbol{\sigma}}_1 = \mathbf{D}_{11}^c \mathbf{E}_{u11}^c \dot{\mathbf{s}}_1 - c_1 \boldsymbol{\delta}_1 \bar{\boldsymbol{\delta}}_2^T \dot{\boldsymbol{\varepsilon}}_2 - \dot{\lambda} c_1 \hat{m}_{1v} \boldsymbol{\delta}_1 \tag{70}$$

where $\mathbf{E}_{u11}^c$ is the (singular) undrained elastic compliance matrix

$$\mathbf{E}_{u11}^c = \mathbf{E}_{11}^c - c_1 \bar{\boldsymbol{\delta}}_1 \bar{\boldsymbol{\delta}}_1^T \tag{71}$$

that resembles the compliance matrix $\mathbf{C}_u^c$ in eqn (39) valid for total stress control.

In order to use the consistency condition eqn (27) it is also necessary to express $\dot{\boldsymbol{\sigma}}_2$ in terms of the control variables $\dot{\mathbf{s}}_1$ and $\dot{\boldsymbol{\varepsilon}}_2$ via the elastic tangent relationship (12). We then obtain, after a few manipulations,

$$\dot{F} = \mathbf{n}_1^T \dot{\boldsymbol{\sigma}}_1 + \mathbf{n}_2^T \dot{\boldsymbol{\sigma}}_2 - H\dot{\lambda} = \phi_u - K_u \dot{\lambda} \leqslant 0 \tag{72}$$

where ϕ_u is the undrained loading function

$$\phi_u = \hat{\mathbf{n}}_1^T \mathbf{D}_{11}^c \mathbf{E}_{u11}^c \dot{\mathbf{s}}_1 + (\hat{\mathbf{n}}_2 - c_1 \hat{n}_{1v} \bar{\boldsymbol{\delta}}_2)^T \dot{\boldsymbol{\varepsilon}}_2 = \hat{\mathbf{n}}_{u1}^T \dot{\mathbf{s}}_1 + \hat{\mathbf{n}}_{u2}^T \dot{\boldsymbol{\varepsilon}}_2 \tag{73}$$

and we have introduced $\hat{n}_{1v} = \boldsymbol{\delta}_1^T \hat{\mathbf{n}}_1$ and the undrained gradients

$$\hat{\mathbf{n}}_{u1} = \hat{\mathbf{n}}_1 - c_1 \hat{n}_{1v} \bar{\boldsymbol{\delta}}_1, \quad \hat{\mathbf{n}}_{u2} = \hat{\mathbf{n}}_2 - c_1 \hat{n}_{1v} \bar{\boldsymbol{\delta}}_2. \tag{74}$$

Furthermore, K_u is the undrained generalized plastic modulus under mixed control

$$K_u = H+\mathbf{n}_2^T\mathbf{E}_{22}^e\mathbf{m}_2+c_1\hat{m}_{1v}\hat{n}_{1v} = K+c_1\hat{m}_{1v}\hat{n}_{1v} \tag{75}$$

where K is the drained modulus as given in eqn (29).

The loading criteria become

$$\dot{\lambda} = \frac{1}{K_u}\phi_u > 0 \quad (\mathrm{P}) \tag{76}$$

$$\phi_u = 0 \quad (\mathrm{N}) \tag{77}$$

$$\phi_u < 0 \quad (\mathrm{E}) \tag{78}$$

and the requirement for unambiguous criteria is obviously that $K_u > 0$.

Inserting $\dot{\lambda}$ from eqn (76) into the expression for $\dot{u}$ in eqn (69), we obtain for $\phi_u > 0$

$$\dot{u} = c_1\left(\boldsymbol{\delta}_1+\frac{1}{K_u}\hat{m}_{1v}\hat{\mathbf{n}}_{u1}\right)^T\dot{\mathbf{s}}_1+c_1\left(\boldsymbol{\delta}_2+\frac{1}{K_u}\hat{m}_{1v}\hat{\mathbf{n}}_{u2}\right)^T\dot{\boldsymbol{\varepsilon}}_2. \tag{79}$$

Finally, combining this expression with the flow rule eqn (65) and inserting in eqns (12), we obtain at plastic loading, i.e. when $F = 0$ and $\phi_u > 0$, the pertinent tangent relationship

$$\begin{bmatrix}\dot{\boldsymbol{\varepsilon}}_1\\ \dot{\mathbf{s}}_2\end{bmatrix} = \begin{bmatrix}\mathbf{E}_{u11} & \mathbf{E}_{u12}\\ \mathbf{E}_{u21} & \mathbf{E}_{u22}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{s}}_1\\ \dot{\boldsymbol{\varepsilon}}_2\end{bmatrix} \tag{80}$$

where

$$\begin{bmatrix}\mathbf{E}_{u11} & \mathbf{E}_{u12}\\ \mathbf{E}_{u21} & \mathbf{E}_{u22}\end{bmatrix} = \begin{bmatrix}\mathbf{E}_{u11}^e & \mathbf{E}_{u12}^e\\ \mathbf{E}_{u12}^e & \mathbf{E}_{u22}^e\end{bmatrix}+\frac{1}{K_u}\begin{bmatrix}\hat{\mathbf{m}}_{u1}\hat{\mathbf{n}}_{u1}^T & \hat{\mathbf{m}}_{u1}\hat{\mathbf{n}}_{u2}^T\\ -\hat{\mathbf{m}}_{u2}\hat{\mathbf{n}}_{u1}^T & -\hat{\mathbf{m}}_{u2}\hat{\mathbf{n}}_{u2}^T\end{bmatrix}. \tag{81}$$

In addition to $\mathbf{E}_{u11}^e$ in (71) we have introduced the undrained elastic moduli

$$\mathbf{E}_{u12}^e = \mathbf{E}_{12}^e-c_1\boldsymbol{\delta}_1\boldsymbol{\delta}_2^T \tag{82}$$

$$\mathbf{E}_{u21}^e = -(\mathbf{E}_{u12}^e)^T = \mathbf{E}_{21}^e+c_1\boldsymbol{\delta}_2\boldsymbol{\delta}_1^T \tag{83}$$

$$\mathbf{E}_{u22}^e = \mathbf{E}_{22}^e+c_1\boldsymbol{\delta}_2\boldsymbol{\delta}_2^T. \tag{84}$$

The undrained plastic flow directions $\hat{\mathbf{m}}_{u1}$ and $\hat{\mathbf{m}}_{u2}$ are defined as in eqn (74), i.e.

$$\hat{\mathbf{m}}_{u1} = \hat{\mathbf{m}}_1-c_1\hat{m}_{1v}\boldsymbol{\delta}_1,\quad \hat{\mathbf{m}}_{u2} = \hat{\mathbf{m}}_2-c_1\hat{m}_{1v}\boldsymbol{\delta}_2. \tag{85}$$

In the case of elastic unloading, $\phi_u \leqslant 0$, we obtain (since $\dot{\lambda} = 0$) from eqn (69)

$$\dot{u} = c_1(\boldsymbol{\delta}_1^T\dot{\mathbf{s}}_1+\boldsymbol{\delta}_2^T\dot{\boldsymbol{\varepsilon}}_2) \tag{86}$$

and from (80)

$$\begin{bmatrix}\dot{\boldsymbol{\varepsilon}}_1\\ \dot{\mathbf{s}}_2\end{bmatrix} = \begin{bmatrix}\mathbf{E}_{u11}^e & \mathbf{E}_{u12}^e\\ \mathbf{E}_{u21}^e & \mathbf{E}_{u22}^e\end{bmatrix}\begin{bmatrix}\dot{\mathbf{s}}_1\\ \dot{\boldsymbol{\varepsilon}}_2\end{bmatrix}. \tag{87}$$

Even in this case of mixed control for undrained behavior it may simply be checked that the special case of total stress control directly follows through the identification $\dot{\mathbf{s}}_1 = \dot{\mathbf{s}}(\dot{\boldsymbol{\varepsilon}}_1 = \dot{\boldsymbol{\varepsilon}})$.

Remark: As mentioned already in the introduction it is not possible to use complete strain control, since we must require $c_t < \infty$ in order for the expressions above to be valid. This means that at least one **normal** strain component must be left as a response variable in order not to overconstrain the strain rates by the undrained condition $\dot{\varepsilon}_v = 0$.

Example of plasticity model: Drucker–Prager's criterion

To illustrate general results, let us consider Drucker–Prager's yield criterion with an (unrealistic) associated flow rule. This criterion, which is pertinent to, typically, non-cohesive granular materials, can be represented as

$$F = II_d - k\sigma_m^2 = 0 \tag{88}$$

where II_d is the second invariant of the effective stress deviator, σ_m is the mean effective stress (as used above), whereas $k > 0$ represents the angle of internal friction and describes the slope of the yield surface in the stress meridian plane.

The stress and strain vectors of interest are represented by their principal values

$$\mathbf{s} = [s_x, s_y, s_z]^T, \quad \boldsymbol{\sigma} = [\sigma_x, \sigma_y, \sigma_z]^T, \quad \boldsymbol{\varepsilon} = [\varepsilon_x, \varepsilon_y, \varepsilon_z]^T \tag{89}$$

which is the relevant situation in a triaxial test. The yield criterion in eqn (88) then takes the explicit form

$$F = (1-k)(\sigma_x+\sigma_y+\sigma_z) - (1+2k)(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x). \tag{90}$$

Let us first consider the situation of complete stress control. The control variables are then all components of s, whereas the response variables are all components of $\boldsymbol{\varepsilon}$ and the pore pressure u. The vector $\boldsymbol{\delta}$, representing Kronecker's delta, becomes simply

$$\boldsymbol{\delta} = [1, 1, 1]^T \tag{91}$$

and the undrained elastic compliance matrix is given as, according to eqn (58),

$$\mathbf{C}_u^e = \frac{1+\nu}{3E}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \tag{92}$$

The gradient of the yield surface, **n**, is given as

$$\mathbf{n} = \begin{bmatrix} 2(1-k)\sigma_x - (1+2k)(\sigma_y+\sigma_z) \\ 2(1-k)\sigma_y - (1+2k)(\sigma_z+\sigma_x) \\ 2(1-k)\sigma_z - (1+2k)(\sigma_x+\sigma_y) \end{bmatrix} \tag{93}$$

and the undrained gradient, $\mathbf{n}_u$, is the deviatoric part of **n**, according to eqn (60),

$$\mathbf{n}_u = \begin{bmatrix} 2\sigma_x - \sigma_y - \sigma_z \\ 2\sigma_y - \sigma_z - \sigma_x \\ 2\sigma_z - \sigma_x - \sigma_y \end{bmatrix} \tag{94}$$

which is the same as for von Mises' criterion.

The undrained modulus, H_u, is given as

$$H_u = H + \frac{12Ek^2}{1-2v}(\sigma_x+\sigma_y+\sigma_z)^2 \tag{95}$$

where the facts that $n_v = -6k$ and $K = E/3(1-2v)$ were used.

Finally, the rate of pore pressure, $\dot{u}$, is given from eqn (51) as

$$\dot{u} = \left[\tfrac{1}{3}\delta - \frac{2Ek}{(1-2v)H_u}(\sigma_x+\sigma_y+\sigma_z)\mathbf{n}_u\right]^T \dot{\mathbf{s}} = \dot{s}_m - \frac{2Ek}{(1-2v)H_u}(\sigma_x+\sigma_y+\sigma_z)\phi_u \tag{96}$$

where ϕ_u is the loading function given in eqn (63).

Let us next consider the case when two stresses are controlled, while one strain component (usually the vertical component in a practical situation) is allowed to change. The control and response variables can then be chosen as

$$[\mathbf{s}_1^T, \boldsymbol{\varepsilon}_2^T]^T = [s_x, s_y, \varepsilon_z]^T, \quad [\boldsymbol{\varepsilon}_1^T, \mathbf{s}_2^T]^T = [\varepsilon_x, \varepsilon_y, s_z]^T \tag{97}$$

and the relevant portions of Kronecker's delta are

$$\delta_1 = [1, 1]^T, \quad \delta_2 = 1. \tag{98}$$

Since the isotropic elastic compliance and stiffness matrices are

$$\mathbf{C}^e = \frac{1}{E}\begin{bmatrix} 1 & -v & -v \\ -v & 1 & -v \\ -v & -v & 1 \end{bmatrix}, \quad \mathbf{D}^e = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v \\ v & 1-v & v \\ v & v & 1-v \end{bmatrix} \tag{99}$$

we obtain the mixed elastic matrix $\mathbf{E}^e$ in eqn (12) as

$$\mathbf{E}^e = \begin{bmatrix} (1-v^2)/E & -v(1+v)/E & -v \\ -v(1+v)/E & (1-v^2)/E & -v \\ v & v & E \end{bmatrix} \tag{100}$$

and, when undrained conditions are imposed,

$$\mathbf{E}_u^e = \begin{bmatrix} (1+v)/2E & -(1+v)/2E & -1/2 \\ -(1+v)/2E & (1+v)/2E & -1/2 \\ 1/2 & 1/2 & 3E/2(1+v) \end{bmatrix}. \tag{101}$$

The latter matrix was obtained from eqns (71) and (82)–(84) with $c_1 = E/2(1-2v)(1+v)$ and

$$\bar{\delta}_1 = \frac{1}{2c_1}[1, 1]^T, \quad \bar{\delta}_2 = 1-2v. \tag{102}$$

The partitioned gradients $\mathbf{n}_1$ and $\mathbf{n}_2$ are obtained, according to eqn (93), as

$$\mathbf{n}_1 = \begin{bmatrix} 2(1-k)\sigma_x - (1+2k)(\sigma_y+\sigma_z) \\ 2(1-k)\sigma_y - (1+2k)(\sigma_z+\sigma_x) \end{bmatrix}, \quad n_2 = 2(1-k)\sigma_z - (1+2k)(\sigma_x+\sigma_y) \tag{103}$$

and the undrained transformed gradients $\hat{\mathbf{n}}_{u1}$ and $\hat{\mathbf{n}}_{u2}$ become, according to eqns (30) and (74),

$$\hat{\mathbf{n}}_{u1} = \frac{3}{2}\begin{bmatrix}\sigma_x - \sigma_y \\ \sigma_y - \sigma_x\end{bmatrix}, \quad \hat{n}_{u2} = \frac{3E}{2(1+v)}(2\sigma_z - \sigma_x - \sigma_y). \tag{104}$$

The generalized undrained plastic modulus, K_u, is obtained from eqn (75) as

$$K_u = H + \frac{12Ek^2}{1-2v}(\sigma_x + \sigma_y + \sigma_z)^2 + \frac{3E}{2(1+v)}(2\sigma_z - \sigma_x - \sigma_y)^2 \tag{105}$$

and it is noted that $K_u > H_u$ (pertinent to stress control) unless $\sigma_x + \sigma_y = 2\sigma_z$, in which case $K_u = H_u$.

Finally, the rate of pore pressure, $\dot{u}$, is given from eqn (79) as

$$\dot{u} = \left(\tfrac{1}{2}\boldsymbol{\delta}_1 + \frac{c_1}{K_u}\hat{n}_{1v}\hat{\mathbf{n}}_{u1}\right)^T \dot{\mathbf{s}}_1 + \left(\frac{E}{2(1+v)} + \frac{c_1}{K_u}\hat{n}_{1v}\hat{n}_{u2}\right)\dot{\varepsilon}_z \tag{106}$$

where $\hat{n}_{1v}$ is given as

$$\hat{n}_{1v} = -4(1+v)k(\sigma_x + \sigma_y + \sigma_z) - (1-2v)(2\sigma_z - \sigma_x - \sigma_y). \tag{107}$$

ASSESSMENT OF STABILITY AND STIFFNESS FOR DRAINED AND UNDRAINED BEHAVIOR

Preliminaries

According to Hill (1958), a sufficient condition for local stability is that the second order incremental work is positive, i.e.

$$\dot{\boldsymbol{\sigma}}^T\dot{\boldsymbol{\varepsilon}} > 0 \tag{108}$$

for all possible choices of the control variables. This condition leads to bounds on the plastic modulus H, which were established by Klisinski *et al.* (1991) for the case of drained behavior i.e. when there are no constraints of the strain or stress components. Because of the kinematic constraint of incompressibility, the situation is somewhat different for undrained behavior.

To simplify the analysis we shall only consider total stress control (since a tangent stiffness formulation pertinent of strain control cannot be established for undrained behavior due to singularity of $\mathbf{C}_u$). It can, in fact, be shown (as for drained behavior) that this evaluation of stability in terms of bounds on H is completely independent of the particular choice of mixed set of control variables. Hence, we shall consider conditions

$$\dot{\boldsymbol{\sigma}}^T\mathbf{C}\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}^T\mathbf{C}^s\dot{\boldsymbol{\sigma}} > 0, \quad \dot{\mathbf{s}}^T\mathbf{C}_u\dot{\mathbf{s}} = \dot{\mathbf{s}}^T\mathbf{C}_u^s\dot{\mathbf{s}} > 0 \tag{109}$$

for all possible control variables in terms of the vectors $\dot{\boldsymbol{\sigma}}$ and $\dot{\mathbf{s}}$, where $\mathbf{C}^s$ and $\mathbf{C}_u^s$ are the symmetric parts of $\mathbf{C}$ and $\mathbf{C}_u$ respectively. (The drained behavior is considered for comparison.) We shall thus establish conditions for which $\mathbf{C}^s$ and $\mathbf{C}_u^s$ are positive definite, if possible.

Apart from stability it is of interest to assess the current stress–strain behavior (in terms of compliance or stiffness) along a certain control path. Again, we only consider stress control, and it is clear that the characteristic behavior is represented by the spectral properties of the (generally non-symmetric) compliance matrices $\mathbf{C}$ and $\mathbf{C}_u$ themselves (and not their symmetric parts as for the stability assessment).

Drained behavior

To assess stability we consider the eigenvalue problem

$$\mathbf{C}^s \mathbf{x}_i = \lambda_i \mathbf{C}^e \mathbf{x}_i, \quad i = 1, 2, \ldots, \dim(\mathbf{C}^e) \tag{110}$$

where, according to eqn (21),

$$\mathbf{C}^s = \mathbf{C}^e + \frac{1}{2H}(\mathbf{m}\mathbf{n}^T + \mathbf{n}\mathbf{m}^T). \tag{111}$$

It has been shown elsewhere, Runesson and Mroz (1989), that the eigenvalues are given as

$$\lambda_{1,2} = 1 + \frac{1}{2H}(\|\mathbf{m}\|_D \|\mathbf{n}\|_D \mp \mathbf{m}^T \mathbf{D}^e \mathbf{n})$$

$$\lambda_k = 1, \quad k = 3, 4, \ldots, \dim(\mathbf{C}^e) \tag{112}$$

where we have introduced the energy norm $\|\mathbf{n}\|_D^2 = \mathbf{n}^T \mathbf{D}^e \mathbf{n}$. (It is noted that the eigenvalues are not listed in order of increasing magnitude.) The condition for positive definite $\mathbf{C}^s$ is $\lambda_1 > 0$, which gives the classical result, e.g. Mroz (1966), Maier and Hueckel (1979), Runesson and Mroz (1989) and Klisinski *et al.* (1991), that $H > H_c$, where H_c is the critical value

$$H_c = \tfrac{1}{2}(\|\mathbf{m}\|_D \|\mathbf{n}\|_D - \mathbf{m}^T \mathbf{D}^e \mathbf{n}) \geqslant 0. \tag{113}$$

In the special case of associated plasticity we obtain $H_c = 0$.

The spectral properties of $\mathbf{C}$ are given from

$$\mathbf{C}\mathbf{y}_i = \mu_i \mathbf{C}^e \mathbf{y}_i, \quad i = 1, 2, \ldots, \dim(\mathbf{C}^e) \tag{114}$$

where $\mathbf{C}$ is still given by eqn (21). It can be shown that the eigenvalues are

$$\mu_1 = 1 + \frac{1}{H}\mathbf{m}^T \mathbf{D}^e \mathbf{n}$$

$$\mu_k = 1, \quad k = 2, 3, \ldots, \dim(\mathbf{C}^e). \tag{115}$$

A limit state is defined by infinite compliance, ($\mu_1 = \infty$), which is obtained when $H = 0$. The eigenvalues of the stiffness matrix $\mathbf{D}$ with respect to the matrix of elastic stiffness moduli $\mathbf{D}^e$ are clearly the inverse values of those in eqn (115). Hence, when $H = 0$ we conclude that $\mathbf{D}$ is singular, as expected.

Undrained behavior

We shall first show that the matrix of elastic moduli $\mathbf{C}_u^e$ defined in eqn (39) is singular corresponding to the eigenvector $\mathbf{z}_1 = \boldsymbol{\delta}$. This result is obtained immediately from

$$\mathbf{C}_u^e \boldsymbol{\delta} = \boldsymbol{\delta} - c_1 \boldsymbol{\delta} c_1^{-1} = 0 \tag{116}$$

where the fact that $\boldsymbol{\delta}^T \boldsymbol{\delta} = 1/c_1$ was used. Moreover, it is simple to show that $\mathbf{z}_k$, $k = 2, 3, \ldots, \dim(\mathbf{C}^e)$, that are defined by the orthogonality condition $\mathbf{z}_k^T \boldsymbol{\delta} = 0$, all correspond to drained elastic response, i.e.

$$\mathbf{C}_u^e \mathbf{z}_k = \mathbf{C}^e \mathbf{z}_k, \quad k = 2, 3, \ldots, \dim(\mathbf{C}^e). \tag{117}$$

For the eigenvalue problem

$$\mathbf{C}_u^e \mathbf{z}_i = v_i \mathbf{C}^e \mathbf{z}_i \tag{118}$$

we thus conclude that the eigenvalues are $v_1 = 0$ and $v_i = 1$ for $i > 2$.

In physical terms the singularity is activated by applying an isotropic total stress, which will only result in increased pore pressure without any deformation. This follows from eqn (49), since with $\dot{\mathbf{s}} = \dot{s}_m \boldsymbol{\delta}$ we obtain

$$\dot{u} = \dot{s}_m. \tag{119}$$

Stress changes along the "elastic" eigenvectors do not produce any change in pore pressure, since $\dot{\mathbf{s}} = \dot{a}\mathbf{z}_k$ gives

$$\dot{u} = c\bar{\boldsymbol{\delta}}^T \mathbf{z}_k \dot{a} = 0 \tag{120}$$

in view of the condition $\mathbf{z}_k^T \bar{\boldsymbol{\delta}} = 0$.

Consider next the eigenvalue problem for elastic–plastic behavior

$$\mathbf{C}_u^* \mathbf{x}_i = \lambda_i \mathbf{C}^e \mathbf{x}_i, \quad i = 1, 2, \ldots, \dim(\mathbf{C}^e) \tag{121}$$

where, according to eqn (48),

$$\mathbf{C}_u^* = \mathbf{C}^e - c\boldsymbol{\delta}\boldsymbol{\delta}^T + \frac{1}{2H_u}(\mathbf{m}_u \mathbf{n}_u^T + \mathbf{n}_u \mathbf{m}_u^T). \tag{122}$$

Even in this case it appears that $\mathbf{x}_1 = \boldsymbol{\delta}$ is an eigenvector corresponding to singularity of $\mathbf{C}_u^*$. This follows from the arguments above for $\mathbf{C}_u^e$ and the fact that $\boldsymbol{\delta}$ is orthogonal to $\mathbf{n}_u$ as well as to $\mathbf{m}_u$ defined in eqns (44) and (48), i.e.

$$\mathbf{n}_u^T \boldsymbol{\delta} = \mathbf{m}_u^T \boldsymbol{\delta} = 0. \tag{123}$$

We shall now consider eigenvectors that are spanned by the vectors $\mathbf{n}_u$ and $\mathbf{m}_u$. This gives the eigenvectors

$$\mathbf{x}_{2,3} = a\mathbf{D}^e \left(\frac{\mathbf{n}_u}{\|\mathbf{n}_u\|_D} \mp \frac{\mathbf{m}_u}{\|\mathbf{m}_u\|_D} \right), \quad a = \text{const.} \tag{124}$$

corresponding to the eigenvalues

$$\lambda_{2,3} = 1 + \frac{1}{2H_u}(\|\mathbf{m}_u\|_D \|\mathbf{n}_u\|_D \mp \mathbf{m}_u^T \mathbf{D}^e \mathbf{n}_u). \tag{125}$$

The formal similarity with the expression in eqn (115) for $\lambda_{1,2}$ pertinent to drained behavior is noteworthy.

The remaining eigenvectors are orthogonal to $\bar{\boldsymbol{\delta}}$ as well as to $\mathbf{n}_u$ and $\mathbf{m}_u$, i.e.

$$\mathbf{x}_k^T \bar{\boldsymbol{\delta}} = \mathbf{x}_k^T \mathbf{n}_u = \mathbf{x}_k^T \mathbf{m}_u = 0, \quad k \geqslant 4 \tag{126}$$

and correspond to $\lambda_k = 1$.

We have thus found all eigenvalues. Since $\lambda_1 = 0$ it is not possible to achieve the condition that $\mathbf{C}_u^*$ is positive definite. However, $\mathbf{C}_u^*$ is positive semi-definite whenever $\lambda_2 > 0$, which gives the condition (similarly to the drained case) $H_u > H_{uc}$ with

$$H_{uc} = \tfrac{1}{2}(\|\mathbf{m}_u\|_D \|\mathbf{n}_u\|_D - \mathbf{m}_u^T \mathbf{D}^e \mathbf{n}_u) \geqslant 0. \tag{127}$$

Now, combining (127) with the definition of H_u in eqn (42) we obtain

$$H_c = \tfrac{1}{2}(\|\mathbf{m}_u\|_D\|\mathbf{n}_u\|_D - \mathbf{m}_u^T\mathbf{D}^e\mathbf{n}_u) - cn_v m_v. \tag{128}$$

Similarly to drained behavior, $H_{uc} = 0$ for associated plasticity. However, in order to achieve this condition it is sufficient to require $\mathbf{m}_u = \mathbf{n}_u$, which behavior is denoted "undrained associativity" and is thus defined by

$$H_{uc} = 0, \quad H_c = -cn_v m_v. \tag{129}$$

The spectral properties of $\mathbf{C}_u$ are given from

$$\mathbf{C}_u \mathbf{y}_i = \mu_i \mathbf{C}^e \mathbf{y}_i, \quad i = 1, 2, \ldots, \dim(\mathbf{C}^e) \tag{130}$$

where $\mathbf{C}_u$ is still given by eqn (49). The eigenvalues are $\mu_1 = 0$ corresponding to isotropic stress change, and $\mu_k = 1$ ($k \geqslant 3$) corresponding to elastic response. The eigenvalue of interest is

$$\mu_2 = 1 + \frac{1}{H_u}\mathbf{n}_u^T\mathbf{D}^e\mathbf{m}_u. \tag{131}$$

A limit state is obtained when $H_u = 0$.

In order to assess the (possible) stabilizing effect from the undrained condition, we shall consider an important special class of material behavior.

Isotropic elasticity and volumetric non-associativity. In the case of isotropic elasticity we have the simplifications $\mathbf{n}_u = \mathbf{n}_d$ and $\mathbf{m}_u = \mathbf{m}_d$. It is quite common that the non-associativity is restricted to the volumetric behavior, i.e. $\mathbf{n}_d = \mathbf{m}_d$. This is clearly a case of undrained associativity defined in eqn (129) and we obtain

$$H_c^{(u)} = -Kn_v m_v \tag{132}$$

where index "u" stands for undrained behavior. It is interesting to note that, for the frictional material defined by $n_v < 0$, the observations made previously are confirmed that dilatant behavior ($m_v < 0$) is stabilizing and implies that $H_c^{(u)} < 0$, whereas contractant behavior ($m_v > 0$) is destabilizing and implies that $H_c^{(u)} > 0$.

The corresponding value of H_c for drained behavior given in eqn (113) is always non-negative

$$H_c^{(d)} = \tfrac{1}{2}[(2G|\mathbf{n}_d|^2 + Kn_v^2)^{1/2}(2G|\mathbf{n}_d|^2 + Km_v^2)^{1/2} - (2G|\mathbf{n}_d|^2 + Kn_v m_v)] \geqslant 0. \tag{133}$$

It is clear that non-associativity has a destabilizing effect for both drained and undrained behavior.

As to the stiffness properties, it appears that eqn (131), that is pertinent to undrained conditions, gives

$$\lambda_{max}^{(u)} = 1 + \frac{2G}{H + Km_v^2}|\mathbf{n}_d|^2 \tag{134}$$

whereas the corresponding value under drained conditions is obtained from eqn (115)

$$\lambda_{max}^{(d)} = 1 + \frac{1}{H}(2G|\mathbf{n}_d|^2 + Kn_v m_v). \tag{135}$$

It is simple to show that $\lambda_{max}^{(u)} \leqslant \lambda_{max}^{(d)}$. Furthermore, $\lambda_{min}^{(u)} = 0$ whereas $\lambda_{min}^{(d)} = 1$, i.e. $\lambda_{min}^{(u)} < \lambda_{min}^{(d)}$. We may thus conclude from the spectral range that the incremental flexibility is smaller for undrained than drained behavior provided the same point on the stress path is considered.

SUMMARY AND CONCLUSIONS

Constitutive relations that are pertinent for modelling the behavior of elastic–plastic material subjected to the constraint of incompressibility, such as undrained behavior of highly impermeable soil, were developed and analyzed in this paper. For pure stress control an "undrained plastic modulus" H_u is defined equivalently to the material plastic modulus for drained response. The pertinent loading criterion, that signals either plastic loading or elastic unloading, was derived. In complete analogy with drained behavior it turns out that a unique response characterization requires that $H_u > 0$.

The discussion was extended to the case of mixed control of a suitable set of total stress and strain components, whereby an "undrained generalized plastic modulus" K_u is substituted for H_u in the assessment of response controllability. Explicit expression of the pore pressure development was also given in terms of the control variables.

For a frictional material that is defined by a cone-type yield surface and which dilates, it was concluded that the condition $H_u > 0$ in stress control is satisfied even when the material undergoes softening, whereas contractant behavior requires hardening. Thus, it seems that dilatancy has a stabilizing effect whereas contractancy is destabilizing, which is also in accordance with previous theoretical findings, e.g. Rice (1975), and with experimental experience, Lade (1988). This important conclusion was also confirmed in the present paper from a more formal discussion of the stability properties, which was based on the spectral properties of the compliance matrix in the undrained (as well as the drained) mode. It was also shown that the most stable situation for undrained behavior is obtained for associated plasticity, in which case the critical value of the hardening modulus is negative for a pressure-dependent yield surface.

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